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# The symmetry approach to higher-dimensional nonlinear equations

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**Abstract.** Conditions for the existence of recursion operators are studied for arbitrary-order scalar partial differential equations with two independent variables. The coefficients of the operator are successively determined by those of the terms with higher degree. For a class of second-order equations, the operators with degree one are completely determined.

## 1. Introduction

It is well known that the Lie–Bäcklund symmetries or the generalized symmetries are important in the study of nonlinear partial differential equations (PDEs). In particular, the existence of an infinite number of such symmetries characterizes the integrability of the PDE. There are several ways to study infinite series of symmetries. Among others, the method of a recursion operator may be the most constructive because it yields the generalized Lax pair for the inverse scattering method for the PDE considered and also a classification of general integrable equations.

For the $(1 + 1)$-dimensional scalar evolution equations, Ibragimov and Shabat [1] presented the problem of determining the recursion operators in a general way. These authors adopted the space $\Im$ of the functions of finite number variables in the jet bundle space $J$ to describe the evolution equation and also the space of formal series of the total differential operator, with the coefficients on $J$, that is the space of the pseudo-differential operator, to describe the recursion operator for generalized symmetries. In a series of papers [2], they derived the necessary and sufficient conditions for the $1 + 1$ scalar evolution equation to have a recursion operator and also a complete classification of the integrable scalar evolution equation with order up to three, i.e. Korteweg–de Vries type equations. For the $(1 + 1)$-dimensional multicomponent evolution equations, the same problem was analysed by Mikhailov *et al* [3]. They presented a list of the two-component integrable evolution equations with order two. For the evolution equations in $(1 + 1)$-dimensional space, if the coefficients of the recursion operator are independent of the explicit 'time' variable, $x^1$, say, the dimension of the differential operator can be reduced to one and hence the algebra of the operators is generated from $\mathrm{D} = \mathrm{d}/\mathrm{d}x^2$ only, where $x^2$ is the 'space' variable.

In this paper, we study the conditions for the existence of the recursion operators for general scalar PDEs with two independent variables. The $(1+1)$-dimensional evolution equation is a special case of our general PDEs. Let $x = \{x^1, x^2\}$ be the independent variable, then the recursion operator is expressed by a formal series of two operators $\mathrm{D}_i = \mathrm{d}/\mathrm{d}x^i$ $(i = 1, 2)$ with the functions on $J$ as coefficients. Our analysis is also valid for the $(1+2)$-dimensional evolution equations, if the derivatives by 'time' are eliminated by means of the evolution equation itself and the recursion operators are assumed to be independent of the explicit 'time' variable.

In section 2, we derive a system of equations for the coefficients of the recursion operators and in section 3, we study the structure of the operator and present a successive relation for the coefficient functions [4]. For a class of second-order PDEs, all the recursion operators with degree one are completely determined in section 4.

## 2. Recursion operators

A PDE of $m$th order for a scalar $u$ with two independent variables $x = \{x^1, x^2\}$ is written in the form

$$f(x, u, u_{(1)}, \ldots, u_{(m)}) = 0 \tag{2.1}$$

where $u_{(k)}$ is the set of $k$th order derivatives

$$u_{(k)} = \{u_{i,j} | i + j = k; 0 \leqslant i, j < \infty\}$$

$$u_{i,j} = \partial^{i+j} u / (\partial x^1)^i (\partial x^2)^j.$$

We call $m$ the order of $f$ and write $\mathrm{O}(f) = m$. In this paper $m$ is assumed to be larger than one, $m \geqslant 2$. The total derivative operators $\mathrm{D}_1$ and $\mathrm{D}_2$ are defined as usual by differential operators on $J$.

Two operators defined by $f$ are introduced:

$$V_f = (\mathrm{D}^{(k)} f) \partial_{(k)} \qquad (0 \leqslant k < \infty)$$

$$f_* = f_{(k)} \mathrm{D}^{(k)} \qquad (0 \leqslant k \leqslant m)$$

where $(k) = (i, j)$,

$$\mathrm{D}^{(k)} = \{\mathrm{D}_1^i \mathrm{D}_2^j | i + j = k, 0 \leqslant i, j < \infty\}$$

$$\partial_{(k)} = \{\partial / \partial u_{i,j} | i + j = k, 0 \leqslant i, j < \infty\}$$

$$f_{(k)} = \{\partial / \partial u_{i,j} f | i + j = k, 0 \leqslant i, j < \infty\}$$

and dummy $(k)$ is summed over the indicated range. We introduce a two-dimensional version of the pseudo-differential operators with commuting set $\mathrm{D}^{(l)} = \{\mathrm{D}_1^{l_1} \mathrm{D}_2^{l_2} | l_1 + l_2 = l, -\infty \leqslant l_1, l_2 \leqslant \infty\}$ subject to a multiplication law

$$\mathrm{D}^{(l)} \circ g = \sum_0^\infty {}_k \binom{l}{k} (\mathrm{D}^{(k)} g) \mathrm{D}^{(l-k)} \tag{2.2}$$

where

$$\sum_0^\infty {}_k = \sum_0^\infty {}_{k_1} \sum_0^\infty {}_{k_2} \qquad \binom{l}{k} = \binom{l_1}{k_1} \binom{l_2}{k_2}$$

and

$$\mathrm{D}^{(l-k)} = \mathrm{D}_1^{l_1-k_1}\mathrm{D}_2^{l_2-k_2}.$$

A recursion operator $L$ for the symmetry $s$ of PDE (2.1) is defined by a formal series

$$L = \sum_{-\infty}^{l}{}_k l_{(k)}\mathrm{D}^{(k)} \tag{2.3}$$

satisfying identically

$$[f_* - V_f, L] = 0 \qquad \text{on } J \tag{2.4}$$

where coefficient $l_{(k)}$ is the set of functions $\{l_{(k_1,k_2)}|k_1 + k_2 = k\}$ on $J$ and

$$\sum_{-\infty}^{l}{}_k = \sum_{-\infty}^{\infty}{}_{k_1}\sum_{-\infty}^{l-k_1}{}_{k_2}.$$

The integer $l$ is called the degree of the formal series and written as $l = \deg L$ if at least one $l_{(l)}$ is non-vanishing. We consider the case in which the coefficients $l_{(k_1,l-k_1)}$ of the leading terms of $L$ are only non-vanishing for finite numbers of $k_1$. Equation (2.4) is an identity of the formal series and the coefficient of the power $\mathrm{D}^{(s)}$ $(s = m+l, m+l-1, \ldots)$ must vanish at each $(s)$. After some calculations including the interchange of the order of summations, (2.4) yields

$$\sum_{s-m}^{l}{}_t\sum_{s-t}^{m}{}_k\left[\binom{k}{k+t-s} f_{(k)}\mathrm{D}^{(k+t-s)}l_{(t)} - \binom{t}{k+t-s} l_{(t)}\mathrm{D}^{(k+t-s)}f_{(k)}\right] = \theta_{(s)}V_f l_{(s)} \tag{2.5}$$

for each $(s) = (s_1, s_2)$ $(s_1 + s_2 \leqslant m + l)$, where $\theta_{(s)} = 1$ for $s$ such that $s_1 + s_2 \leqslant l$, $= 0$ for $l < s_1 + s_2 \leqslant l + m$, $(t) = (t_1, t_2)$ is summed in the domain $D_T$; $t_1 + t_2 \geqslant s_1 + s_2 - m$, $t_1 \geqslant s_1 - m$, $t_2 \geqslant s_2 - m$, $t_1 + t_2 \leqslant l$ on the $t$-plane and $(k) = (k_1, k_2)$ is summed in the domain $D_K$; $k_1 \geqslant \max\{0, s_1 - t_1\}$, $k_2 \geqslant \max\{0, s_2 - t_2\}$, $k_1 + k_2 \leqslant m$ on the $k$-plane.

## 3. Equations for $l_{(s)}$

In this section, we study the structure of (2.5) and present equations to determine $l_{(s)}$ successively for arbitrarily fixed $m = \mathrm{O}(f)$ and $l = \deg L$. From the shape of the domains of the summations on $t$ and $k$, it is convenient to choose $s_1$ and $s_1 + s_2$ as independent parameters.

First we note that in the case $s_1 + s_2 = m + l$, (2.5) is trivially satisfied since for commutators the formal leading-order terms always vanish.

*3.1. Equations for* $s_1 + s_2 = m + l - 1$

In this case, the summations on $t$ are taken along the lines $t_1 + t_2 = l$ and $t_1 + t_2 = l - 1$ in the $t$ plane. However, the summations along the line $t_1 + t_2 = l - 1$ are shown to vanish identically and (2.5) takes the form

$$\sum_{s_1-m}^{s_1+1} {}_{t_1}\Gamma_1(s_1, t_1) l(t_1, l - t_1) = 0 \tag{3.1}$$

where $\Gamma_1(s, t)$ is a differential operator

$$\Gamma_1(s,t) = (s - t + 1) f_{(s-t+1,m-s+t-1)} \mathrm{D}_1 - t[\mathrm{D}_1 f_{(s-t+1,m-s+t-1)}] + (m - s + t) f_{(s-t,m-s+t)} \mathrm{D}_2 - (l - t)[\mathrm{D}_2 f_{(s-t,m-s+t)}].$$

For each $s_1$, (3.1) contains $m + 2$ independent $l_{(t_1,l-t_1)}$, $\{l_{(s_1-m,l+m-s_1)}, l_{(s_1-m+1,l+m-s_1-1)}, \ldots, l_{(s_1+1,l-s_1-1)}\}$ and the term $\Gamma_1(s_1, t_1) l_{(t_1,l-t_1)}$ is presented by a dot on the $t_1 - s_1$ plane in figure 1, where $l = 3$ and $m = 2$ are chosen for the definiteness and the dots with a common $s_1$ represent all the terms in (3.1). An unfilled circle at $(0, -1)$ denotes that $\Gamma_1(-1, 0) l(0, l)$ does not appear in (3.1) with $s_1 = -1$ while an unfilled circle at $(l, l + m)$ means the lack of $\Gamma_1(l + m, l) l_{(l,0)}$ in (3.1) with $s_1 = l + m$. A vertical alley of the dots represents terms with common $l_{(t_1,l-t_1)}$ among equations (3.1) with different $s_1$. It is noted that on the boundary lines $s_1 = t_1 - 1$ and $s_1 = t_1 + m$, $\Gamma_1(s_1, t_1)$ is not the differential operator but simply the multiplication of functions $\Gamma_1(s_1, s_1 + 1) = -(s_1 + 1)\mathrm{D}_1 f_{(0,m)}$ and $\Gamma_1(s_1, s_1 - m) = -(l + m - s_1)\mathrm{D}_2 f_{(m,0)}$, respectively. Hence, if $f_{(0,m)}$ and $f_{(m,0)}$ are not constants, which we assume hereafter, the only possible case where the finite numbers of $l_{(t_1,l-t_1)}$ are non-zero is that in which the set (3.1) with $s_1 = 0, 1, \ldots, l + m - 1$ is closed with respect to the set $l_{(t_1,l-t_1)}$ $(t_1 = 0, 1, \ldots, l)$ and the other $l_{(t_1,l-t_1)}$ $(t_1 < 0$ and $t_1 > l)$ vanish identically. Thus for a prescribed $f$, we have an overdetermined system of $l + 1$ unknowns $l_{(t_1,l-t_1)}$

$$\sum_{s_1-m}^{s_1+1} {}_{t_1}\Gamma_1(s_1, t_1) l_{(t_1,l-t_1)} = 0 \qquad (0 \leqslant s_1 \leqslant l + m - 1) \tag{3.2}$$

in the lozenge cut by $t_1 = 0$ and $t_1 = l$ and

$$l_{(t_1,l-t_1)} = 0 \qquad (t_1 < 0 \text{ and } t_1 > l). \tag{3.3}$$

*3.2. Equations for* $s_1 + s_2 = m + l - i$ $(i \geqslant 2)$

The summation of $t$ is taken on the trapezoidal region defined by $t_1 + t_2 \geqslant l - i, t_1 \geqslant s_1 - m, t_2 \geqslant s_2 - m$ and $t_1 + t_2 \leqslant l$ on the $t$ plane. We can show that the summation along the line $t_1 + t_2 = l - i$ vanishes identically. Since the unknown $l_{(t_1,t_2)}$ with $t_1 + t_2 = l - k$ $(k \geqslant 1)$ are all located only on the line $t_1 + t_2 = l - k$, the summation of the terms including $l_{(t_1,t_2)}$ with $t_1 + t_2 = l - i + 1$ and those with $t_1 + t_2 > l - i + 1$ can be separated

$$\sum_{s_1-m}^{s_1+1} {}_{t_1}\Gamma_i(s_1, t_1) l_{(t_1,l-t_1+1-i)} = \Lambda_i \qquad (0 \leqslant s_1 \leqslant l + m - i) \tag{3.4($s_1$)}$$

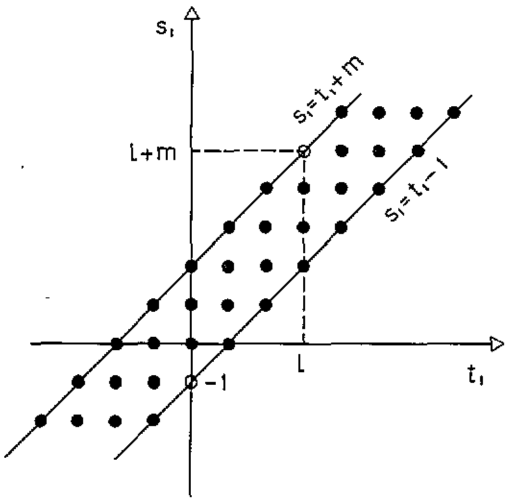


**Figure 1.** $t_1 - s_1$ plane presenting the terms in (3.1). The figure shows the $l = 3$ and $m = 2$ case.

where

$$\Gamma_i(s,t) = (s-t+1) f_{(s-t+1,m-s+t-1)} \mathrm{D}_1 - t[\mathrm{D}_1 f_{(s-t+1,m-s+t-1)}] \\ + (m-s+t) f_{(s-t,m-s+t)} \mathrm{D}_2 - (l+1-i-t)[\mathrm{D}_2 f_{(s-t,m-s+t)}] \qquad (3.5)$$

and $\Lambda_i$ contains $l_{(t_1,l-t_1+1-j)}$ with $j$ only in the interval $1 \leqslant j < i$.

Suppose that all $l_{(t_1,t_2)}$ with $t_1+t_2 > l-i+1$ for fixed $s_1$ and $i$ are given, then $(3.4(s_1))$ is a differential equation for $m+2$ unknowns $\{l_{t_1,l-t_1+1-i)} | s_1 - m \leqslant t_1 \leqslant s_1+1\}$ and has solutions for a class of PDE $f = 0$.

As for the case of $s_1 + s_2 = m + l - 1$ discussed in section 3.1, we consider the condition that only finite numbers of $l_{(t_1,t_2)}$ for $t_1 \geqslant 0$ and $t_2 \geqslant 0$ are non-vanishing. For a fixed $i$, each term of the left-hand side of $(3.4(s_1))$ is presented in figure 2 by a dot with the corresponding coordinate $(t_1, s_1)$ on the line $s_1 =$ constant. Two unfilled circles show the absence of $\Gamma_i(s_1,t_1) l_{(t_1,l-t_1+1-i)}$ at the corresponding values of $s_1$ and $t_1$. Thus the only possible case for a finite number of $\Gamma_i l$ to be non-vanishing is that $\Gamma_i l$ are vanishing for all $t_1$ such that $t_1 > l+1-i$ or $t_1 \leqslant -1$

$$l_{(t_1,l-t_1+1-i)} = 0 \qquad (t_1 < 0 \text{ and } t_1 > l+1-i) \qquad (3.6)$$

and that for $i$ in the range $2 \leqslant i \leqslant l+1$ we have $l+m+1-i$ equations

$$\sum_{s_1-m}^{s_1+1} {}_{t_1}\Gamma_i(s_1,t_1) l_{(t_1,l-t_1+1-i)} = \Lambda_i \qquad (0 \leqslant s_1 \leqslant l+m-i) \qquad (3.7)$$

for $l+1-i$ functions $l_{(t_1,l-t_1+1-i)}$ $(0 \leqslant t_1 \leqslant l+1-i)$ in the lozenge cut by the lines $t_1 = 0$ and $t_1 = l+1-i$. The lozenge reduces to a section of the line $t_1 = 0$ for $i = l+1$. Thus there are $m$ equations to determine one unknown function $l_{(0,0)}$. For $i$ larger than $l+1$, the open circle on the upper boundary moves to the left-hand side of the $s_1$ axis and hence $L$ must be a semi-infinite series of the negative powers of $\mathrm{D}^{(s)}$. In this case,

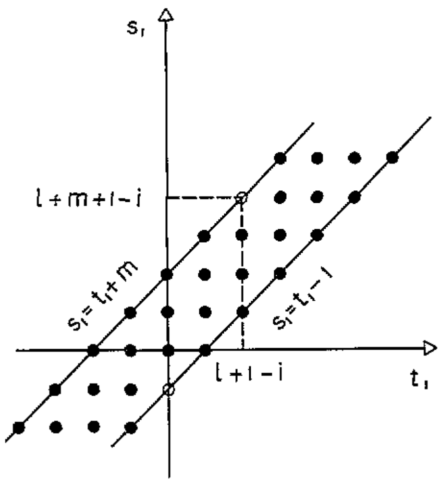


Figure 2. $t_1 - s_1$ plane presenting the terms in the right-hand side of (3.4($s_1$)). The figure shows the $l = 3$, $m = 2$ and $i = 2$ case.

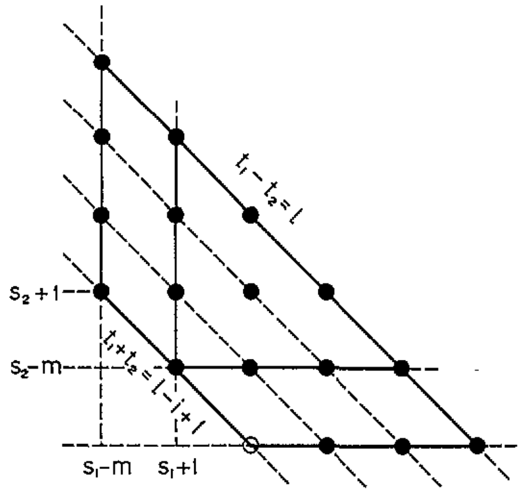


Figure 3. $t_1 - t_2$ plane showing the recursive structure of $l_{(t_1,t_2)}$ in (3.4($s_1$)).

however, (3.4($s_1$)) is a system of algebraic equations for $l_{(t_1-m,l+m+1-i-t_1)}$ or $l_{(t_1+1,l-i-t_1)}$ on the boundary lines and we can always determine unknown $l$s with a certain number of $l$s being arbitrary.

Let us show in figure 3 the recursive structure of (3.4($s_1$)) with respect to $l_{(t_1,l-t_1+1-i)}$. For a fixed $i$, we assume that all $l_{(t_1,t_2)}$ with $t_1 + t_2 > l - i + 1$ are known and so $\Lambda_i$ in (3.4($s_1$)) is a known function. We also assume that (3.4($s_1$)) has solutions for any $s_1$ and a class of $f$. Then increasing or decreasing $s_1$ by one in (3.4($s_1$)), we obtain a new equation where $l_{(s_1+2,l-s_1-1-i)}$ for (3.4($s_1+1$)) or $l_{(s_1-1-m,l-s_1+2+m-i)}$ for (3.4($s_1-1$)) only is unknown and from our assumptions these unknowns are determined from (3.4($s_1 \pm 1$)). These processes make us determine all $l_{(t_1,l-t_1+1-i)}$ on the line $t_1 + t_2 = l + 1 - i$ and from the reduction on $i$ we can obtain, in principle, all $l_{(t_1,t_2)}$s with $t_1 + t_2 \leqslant l - i + 1$ ($i \geqslant 2$).

## 4. Example for $\mathrm{O}(f) = 2$ and $\deg L = 1$

The analysis of (3.7) for general $l = \deg L$ and $m = \mathrm{O}(f)$ is laborious work and has not yet been completed. Here we restrict ourselves to the case $l = 1$ and $m = 2$ and assume that the function $f$ defining the PDE and the coefficients $l_{(k)}$s for the operator $L$ are functions of $X = u_{20}$, $Y = u_{11}$ and $Z = u_{02}$ only. Furthermore, $L$ is assumed to have only the leading terms. Then, we will exhaust all the solutions of the pair $f$ and $L$ to (2.4). Following the discussions in section 3.2, $L$ with (3.3) has the form

$$L = l_{(10)}\mathrm{D}_1 + l_{(01)}\mathrm{D}_2 \tag{4.1}$$

and

$$f_* = f_X\mathrm{D}_1^2 + f_Y\mathrm{D}_1\mathrm{D}_2 + f_Z\mathrm{D}_2^2 \tag{4.2}$$

where $f_X = f_{(2,0)}$, $f_Y = f_{(1,1)}$ and $f_Z = f_{(0,2)}$. Since the forms of the operators are restricted, it is more straightforward to consider the algebra of (4.1), (4.2) and (2.4) directly rather than to deal with (3.2).

Let $S_i$ ($i = 1, 2$) be the linear space of the truncated operators as (4.1) and (4.2) with the degree 1 and 2 respectively. We divide the totality of $L$s into two classes: (A) $L$ commutes only with constant multiples of $L$, $M = CL$ ($C$ = constant); and (B) $L$ has commuting element $M \in S_1$ linearly independent of $L$. Operators in $S_2$ may be expanded by operators in $S_1$. We choose, as the basis of $S_1$, $\{L, \mathrm{D}_1\}$ for case A and $\{L, M\}$ for case B and expand $N \in S_2$ in terms of these bases. The coefficients in the expansion of $f_* \in S_2$ and $L \in S_1$ are determined from the conditions obtained from (2.4).

### 4.1. Case A

*Proposition 1.* Let $L \in S_1$ commute only with $M = CL$ ($C$ = constant). Then the condition

$$\deg[L, N] \leqslant 1 \qquad (N \in S_2) \tag{4.3}$$

implies that there exists a constant $C'$ such that $\deg(N - C'L^2) \leqslant 1$.

*Proof.* In terms of the basis $\{L, D\}$ ($D = \mathrm{D}_1$), $N$ may be written in the form

$$N = L(\xi L + \eta D) + \zeta D^2 + N_1$$

where $N_1 \in S_1$, and $\xi$, $\eta$ and $\zeta$ are functions of $X$, $Y$ and $Z$. If $\zeta$ is not identically zero, we can put $\zeta = \pm 1$ by rewriting $(\pm\zeta)^{1/2} D$ as $D$. Since the coefficients of $D^2$ and $DL$ in $[L, N]$ must vanish, we have $[L, D] = \tau L$ and $L\eta \pm 2\tau = 0$ for some function $\tau$. Then it is easy to see that $D' = D \pm \eta L/2$ commutes with $L$, which contradicts the assumption in proposition 1 and hence $\zeta = 0$. From $N = L\overline{L} + N_1$ ($\overline{L} \in S_1$) and (4.3), one has $[L, \overline{L}] = 0$, which gives $\overline{L} = C'L$ ($C'$ = constant). □

Since (2.4) yields $\deg([f_*, L]) \leqslant 1$, we may put, without loss of generality,

$$f_* = L^2 + N_1 \qquad (N_1 \in S_1) \tag{4.4}$$

and study further conditions on $f$.

For $f_*$ and $L$ given by (4.1) and (4.2), the leading terms in (4.4) yield the conditions

$$f_X = (l_{(10)})^2 \qquad f_Y = 2l_{(10)}l_{(01)} \qquad f_Z = (l_{(01)})^2.$$

Let us choose the branch of the roots as

$$l_{(10)} = f_X^{1/2} \qquad l_{(01)} = \pm f_Z^{1/2} \tag{4.5}$$

and obtain a necessary condition

$$f_Y = \pm 2 f_X^{1/2} f_Z^{1/2}. \tag{4.6}$$

Then (4.1), (4.2), (4.5), (4.6) and (2.4) reduce to 20 equations for $f$ as the condition that (2.4) is an identity with respect to the variables of order 3, $u_{(3)}$. Among these 20 equations, 16 equations are found to hold trivially by REDUCE, provided the rest, four equations, are satisfied. Eventually we find

$$f = (1/R)(XZ - Y^2) \qquad (R = C_1^2 X + 2C_1C_2Y + C_2^2 Z) \tag{4.7}$$

where $C_1$ and $C_2$ are arbitrary constants. The operator $L$ is given by (4.1), where $l_{(10)}$ and $l_{(01)}$ are given from (4.5) and (4.7) as

$$l_{(10)} = (1/R)(C_1Y + C_2Z) \qquad l_{(01)} = -(1/R)(C_1X + C_2Y).$$

There is an ambiguity in the transformation $(X, Y, Z) \to (X + X_0, Y + Y_0, Z + Z_0)$ where $X_0$, $Y_0$ and $Z_0$ are arbitrary constants. Thus the general form of $f$ and $L$ depends on five parameters.

*4.2. Case B*

*Proposition 2.* Let $L \in S_1$ and $M \in S_1$ be linearly independent and satisfy $[L, M] = 0$ and let $P$ be the operator such that $\deg P = 2$ and $\deg [L, P] \leqslant 1$ and let $L$, $M$ and $P$ have the leading terms with the coefficients depending only on $X$, $Y$ and $Z$. Then $P$ has the leading terms in the form

$$P = m_1 L^2 + m_2 LM + m_3 M^2 + \cdots$$

where $m_i$ $(i = 1, 2, 3)$ are constants.

*Proof.* Let $m$ be function of $X$, $Y$ and $Z$. Then $Lm = 0$ implies $m$ = constant since, in $Lm$, coefficients of $\mathrm{D}_1 X$, $\mathrm{D}_2 X$ and so on must vanish. A general operator of $\deg P = 2$ can be presented as

$$P = m_1 L^2 + m_2 LM + m_3 M^2 + \cdots$$

where $m_i$ $(i = 1, 2, 3)$ are functions of $X$, $Y$ and $Z$. From the condition $\deg[L, P] \leqslant 1$ and

$$[L, P] = (Lm_1)L^2 + (Lm_2)LM + (Lm_3)M^2 + \cdots$$

one has $Lm_i = 0$ and hence $m_i$ = constant. □

We give $L$ and $M$ in proposition 2 in explicit forms.

*Proposition 3.* Let $\overline{L}_1 \in S_1$ and $\overline{L}_2 \in S_1$ be linearly independent, and satisfy $[\overline{L}_1, \overline{L}_2] = 0$, and have the leading terms with the coefficients depending only on $X$, $Y$ and $Z$, then $\overline{L}_1$ and $\overline{L}_2$ are the linear combinations of $L_1$ and $L_2$ given by

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} X & -\overline{Y} \\ -Y & Z \end{pmatrix} \begin{pmatrix} \mathrm{D}_1 \\ \mathrm{D}_2 \end{pmatrix} \qquad (\Delta = XZ - Y\overline{Y}, \overline{Y} = Y + C, C = \text{constant}) \quad (4.8)$$

except for the ambiguity of the transformation $X \to X + X_0$, $Y \to Y + Y_0$ and $Z \to Z + Z_0$ where $X_0$, $Y_0$ and $Z_0$ are constants.

The proof is given in the appendix.

Let us expand the recursion operator $L \in S_1$ and $f_* \in S_2$ in terms of $L_1$ and $L_2$,

$$L = n_1 L_1 + n_2 L_2 \tag{4.9}$$

where $n_1$ and $n_2$ are arbitrary constants and

$$f_* = m_1 L_1^2 + m_2 L_1 L_2 + m_3 L_2^2 + \cdots \tag{4.10}$$

where the $m_i$ $(i = 1, 2, 3)$ are also constants by the condition $\deg [L, f_*] \leqslant 1$ and the proposition 2. Substituting (4.8) into (4.10), we obtain necessary conditions for $f$ as

$$f_X = \frac{1}{\Delta^2}(m_1 Z^2 - m_2 YZ + m_3 Y^2)$$

$$f_Y = \frac{1}{\Delta^2}[-2m_1 \overline{Y} Z + m_2(XZ + Y\overline{Y}) - 2m_3 XY]$$

$$f_Z = \frac{1}{\Delta^2}(m_1 Y\overline{Y} - m_2 X\overline{Y} + m_3 X^2).$$

These equations are integrable for $f$ iff $Y = \overline{Y}$, i.e. $C = 0$ to lead

$$f = \frac{1}{\Delta}(-m_1 Z + m_2 Y - m_3 X) + m_4 \qquad (\Delta = XZ - Y^2) \tag{4.11}$$

with arbitrary values of the constants $m_i$ ($i = 1, 2, 3, 4$). It is confirmed that (2.4) is satisfied at every degree for (4.9) and (4.11) with constants $n_i$ and $m_i$ arbitrary.

Thus one has (4.11) as the PDE $f = 0$ for case B. Since (4.11) also admits the transformation $(X, Y, Z) \to (X + X_0, Y + Y_0, Z + Z_0)$ ($X_0, Y_0, Z_0$ = constant), (4.11) eventually has seven parameters while its recursion operator (4.9) depends on five parameters. Khabirov [5] studied a general second-order equation admitting symmetry in a different context but, in particular, the recursion operator was not discussed.

As we have mentioned in the introduction, (4.1) is also a recursion operator for the $(1+2)$-dimensional evolution equation

$$u_t - f = 0$$

where $u_t$ is a time derivative of $u$.

## Appendix

Here we give a theorem and a proof of proposition 3 in the text.

*Theorem.* Let $L_i = l_{ij} \mathrm{D}_j$ ($i, j = 1, 2$) be linearly independent first-degree operators such that $2 \times 2$ matrix $l = \{l_{ij}\}$ are functions of $x_i$ ($i = 1, 2$) and are invertible. Then the equation $[L_1, L_2] = 0$ yields the general form of $l^{-1}$ as

$$\{l^{-1}\}_{ij} = \mathrm{D}_i \varphi_j \tag{A.1}$$

where $\varphi_j$ ($j = 1, 2$) are functions of $\{x_i\}$ such that the matrix $\{\mathrm{D}_i \varphi_j\}$ is invertible.

*Proof.* Since $l$ is invertible we have $\mathrm{D}_i = (l^{-1})_{ij} L_j$. Let $\mathrm{d} \equiv \mathrm{d}x^i \mathrm{D}_i$ be the exterior derivative, then $\mathrm{d}^2 = \mathrm{d}x^i \mathrm{d}x^j \mathrm{D}_i \mathrm{D}_j = 0$ implies $[\mathrm{D}_i, \mathrm{D}_j] = 0$. On the other hand,

$$\begin{aligned}
\mathrm{d}^2 &= \mathrm{d}x^i\, \mathrm{d}x^j \mathrm{D}_i l_{jl}^{-1} L_l \\
&= \mathrm{d}x^i \mathrm{d}x^j [(\mathrm{D}_i l_{jl}^{-1}) L_l + l_{jl}^{-1} \mathrm{D}_i L_l] \\
&= \tfrac{1}{2} \mathrm{d}x^i\, \mathrm{d}x^j [(\mathrm{D}_i l_{jl}^{-1} - \mathrm{D}_j l_{il}^{-1}) L_l + l_{jl}^{-1} l_{ik}^{-1} [L_k, L_l]] \\
&= [\mathrm{d}(l_{jl}^{-1} \mathrm{d}x^j)] L_l
\end{aligned}$$

by virtue of $[L_k, L_l] = 0$. Thus one has $\mathrm{d}(l_{jl}^{-1} \mathrm{d}x^j) = 0$, or $l_{jl}^{-1} \mathrm{d}x^j = \mathrm{d}\varphi_l = \mathrm{D}_j \varphi_l \mathrm{d}x^j$, for some function $\varphi_l$, i.e. $l_{jl}^{-1} = \mathrm{D}_j \varphi_l$. □

*Proof of proposition 3.* Let $\{\varphi_j\}$ be functions on $J$ and $L_i \in S_1$, and $L_i$ have terms with the coefficients depending only on $X$, $Y$ and $Z$, then from (A.1), $\varphi_j$ must be a linear function of $u_{(1)}$ and $\{x^i\}$. Let $B = \{B_{ij}\}$ and $C = \{C_{ij}\}$ be constant $2 \times 2$ matrices and $B$ be invertible, then we can set

$$\frac{\partial \varphi_i}{\partial u_{10}} = (B^{-1})_{1i} \qquad \frac{\partial \varphi_i}{\partial u_{01}} = (B^{-1})_{2i}$$

and

$$\frac{\partial \varphi_j}{\partial x^i} = (CB^{-1})_{ij}.$$

Then equation (A.1) gives

$$l^{-1} = (U + C)B^{-1}$$

where

$$U = \begin{pmatrix} u_{20} & u_{11} \\ u_{11} & u_{02} \end{pmatrix} = \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix}$$

and $l = B(U + C)^{-1}$. Without loss of generality, we may set $B = \{\delta_{ij}\}$ and

$$l = \frac{1}{|U + C|} \begin{pmatrix} X + C_{22} & -Y - C_{12} \\ -Y - C_{21} & Z + C_{11} \end{pmatrix}.$$

A suitable transformation of $X$, $Y$ and $Z$ reduces $l$ to (4.8) for $L_i$. □

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